

On the Cycle Space of a Random Graph

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Sept 2016

Abstract

Write $\mathcal{C}(G)$ for the cycle space of a graph G , $\mathcal{C}_\kappa(G)$ for the subspace of $\mathcal{C}(G)$ spanned by the copies of the κ -cycle C_κ in G , \mathcal{T}_κ for the class of graphs satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$, and \mathcal{Q}_κ for the class of graphs each of whose edges lies in a C_κ . We prove that for every odd $\kappa \geq 3$ and $G = G_{n,p}$,

$$\max_p \Pr(G \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \rightarrow 0;$$

so the C_κ 's of a random graph span its cycle space as soon as they cover its edges. For $\kappa = 3$ this was shown in [6].

1 Introduction

An issue of considerable interest in combinatorics over the last few decades has been the extent to which various standard facts, for instance the classic theorems of Turán, Ramsey and Szemerédi, remain true in a “sparse random” setting. Thus, for example, one may ask for which p a given (deterministic) assertion regarding the complete graph K_n is likely to hold in the (“Bernoulli”) random graph $G_{n,p}$. Our main result follows this theme.

Our underlying deterministic statement is Proposition 1.1 below, for which we need a few definitions. The *edge space* of a graph G , denoted $\mathcal{E}(G)$, is the vector space $\mathbb{F}_2^{E(G)}$. Its elements are naturally identified with

*Department of Mathematics, Rutgers University, Piscataway, NJ. Supported by the U.S. Department of Homeland Security under Grant Award 2012-ST-104-000044. The views and conclusions contained in this document are those of the authors and should not be interpreted as necessarily representing the official policies, either express or implied, of the U.S. Department of Homeland Security.

†Department of Mathematics, Rutgers University, Piscataway, NJ. Supported by the National Science Foundation under Grant Awards DMS1201337 and DMS1501962.

the (spanning) subgraphs of G . The *cycle space* of G , denoted $\mathcal{C}(G)$, is the subspace of $\mathcal{E}(G)$ generated by the (indicators of) cycles of G (see e.g. [7, Sec. 1.9] for an exposition). For a fixed graph H , the *H-space* of G is the subspace of $\mathcal{E}(G)$ generated by the copies of H in G ; this will be denoted $\mathcal{C}_H(G)$, or simply $\mathcal{C}_\kappa(G)$ if $H = C_\kappa$ (the κ -cycle or κ -gon).

Proposition 1.1. *If $\kappa \geq 3$ is odd, then for any $n \geq \kappa$, $\mathcal{C}_\kappa(K_n) = \mathcal{C}(K_n)$.*

Of course for even κ , $\mathcal{C}_\kappa(G)$ is at most the space spanned by *even* cycles. Below, in Theorem 1.5, we will characterize $\mathcal{C}_H(K_n)$ for any fixed H and large enough n .

Assuming κ is odd, when, in terms of p ($= p(n)$), are the κ -gons of $G_{n,p}$ likely to span its cycle space? Let \mathcal{T}_κ be the class of graphs G satisfying $\mathcal{C}_\kappa(G) = \mathcal{C}(G)$ and let \mathcal{Q}_κ be the class of nonempty graphs each of whose edges lies in a copy of C_κ . For any G , it's easy to see that $G \notin \mathcal{T}_\kappa$ unless every edge of G that lies in a cycle in fact lies in a κ -gon. On the other hand, if $p > (1 + \Omega(1)) \log n/n$ then w.h.p.¹ every edge of $G_{n,p}$ does lie in a cycle (e.g. [13, p. 105]). So for such p , $G_{n,p} \in \mathcal{T}_\kappa$ w.h.p. at least requires $G_{n,p} \in \mathcal{Q}_\kappa$ w.h.p., and we should first understand when this is true. Let

$$p_\kappa^* = p_\kappa^*(n) = [(\kappa/(\kappa-1))n^{-(\kappa-2)} \log n]^{1/(\kappa-1)} \quad (1)$$

(where we always use \log for \ln). Note \mathcal{Q}_κ is not an *increasing* property—that is, it is not preserved by adding edges. Nonetheless, p_κ^* is a *sharp threshold* for \mathcal{Q}_κ , in the sense that:

Lemma 1.2. *For any fixed $\kappa \geq 3$ and $\varepsilon > 0$,*

$$\Pr(G_{n,p} \in \mathcal{Q}_\kappa) \rightarrow \begin{cases} 0 & \text{if } p < (1 - \varepsilon)p_\kappa^*, \\ 1 & \text{if } p > (1 + \varepsilon)p_\kappa^*. \end{cases} \quad (2)$$

(Throughout the paper limits are taken as $n \rightarrow \infty$.) We prove this routine observation in Section 4. The cases in (2) are called the *0-statement* and the *1-statement* (respectively).

Given Lemma 1.2, one might hope that p_κ^* is also a sharp threshold for \mathcal{T}_κ , and it essentially is, but for a small glitch in the 0-statement: for $p < (1 - \Omega(1))/n$, we have $\lim \Pr(G_{n,p} \in \mathcal{T}_\kappa) > 0$ for the silly reason that the probability of having no cycles at all is (asymptotically) positive (see e.g. [18, Thm. 1]). Thus we will show:

¹W.h.p. (“with high probability”) means with probability tending to 1 as $n \rightarrow \infty$.

Theorem 1.3. *For any fixed odd $\kappa \geq 3$ and $\varepsilon > 0$,*

$$\Pr(G_{n,p} \in \mathcal{T}_\kappa) \rightarrow \begin{cases} 0 & \text{if } (1 - o(1))/n < p < (1 - \varepsilon)p_\kappa^*, \\ 1 & \text{if } p > (1 + \varepsilon)p_\kappa^*. \end{cases}$$

We actually prove the following stronger statement (see Section 4 for “stronger”), which says that edges not in κ -gons are the obstruction to \mathcal{T}_κ in a precise sense. This is our main result.

Theorem 1.4. *For any fixed odd $\kappa \geq 3$,*

$$\max_p \Pr(G_{n,p} \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \rightarrow 0; \quad (3)$$

equivalently,

$$\forall p = p(n), \quad \Pr(G_{n,p} \in \mathcal{Q}_\kappa \setminus \mathcal{T}_\kappa) \rightarrow 0. \quad (4)$$

(The (trivial) equivalence is given by the observation that (4) holds iff it holds when, for each n , $p = p(n)$ is a value achieving the maximum in (3) (and in this case the two statements are the same).)

Theorems 1.3 and 1.4 for $\kappa = 3$ were proved in [6]; even the former had been open and of interest, being the first unsettled case of a conjecture of M. Kahle (see [14, 15]) on the homology of the clique complex of $G_{n,p}$. Perhaps surprisingly, the argument of [6] does not extend to $\kappa \geq 5$, though, as discussed below, it does share a starting point with what we do here.

What happens if we replace the C_κ of Proposition 1.1 by some other graph? With $\mathcal{D}(G) = \{D \in \mathcal{E}(G) : |D| \equiv 0 \pmod{2}\}$, the proposition generalizes neatly:

Theorem 1.5. *For any graph H with at least one edge and n large enough with respect to H ,*

$$\mathcal{C}_H(K_n) = \begin{cases} \mathcal{C}(K_n) & \text{if } H \text{ is Eulerian and } |H| \text{ is odd,} \\ \mathcal{C}(K_n) \cap \mathcal{D}(K_n) & \text{if } H \text{ is Eulerian and } |H| \text{ is even,} \\ \mathcal{E}(K_n) & \text{if } H \text{ is not Eulerian and } |H| \text{ is odd,} \\ \mathcal{D}(K_n) & \text{if } H \text{ is not Eulerian and } |H| \text{ is even.} \end{cases} \quad (5)$$

Here $|H| = |E(H)|$ and “Eulerian” means degrees are even, but not that the graph is necessarily connected. Of course the left-to-right containments ($\mathcal{C}_H(K_n) \subseteq \mathcal{C}(K_n)$ and so on) are obvious.

The natural value of $\mathcal{C}_H(G)$, which we will denote $\mathcal{W}_H(G)$, is then what one gets by replacing K_n by G in the appropriate expression on the r.h.s. of (5); e.g. for $H = C_\kappa$,

$$\mathcal{W}_H(G) = \begin{cases} \mathcal{C}(G) & \text{if } \kappa \text{ is odd,} \\ \mathcal{C}(G) \cap \mathcal{D}(G) & \text{if } \kappa \text{ is even.} \end{cases} \quad (6)$$

(We could instead set $\mathcal{W}_H(G) = \mathcal{E}(G) \cap \mathcal{C}_H(K_n)$, which by Theorem 1.5 is the same for all but a few values of n .) So we are interested in understanding when $G_{n,p}$ is likely to lie in

$$\mathcal{T}_H := \{G : \mathcal{C}_H(G) = \mathcal{W}_H(G)\}.$$

(Again, $\mathcal{C}_H(G) \subseteq \mathcal{W}_H(G)$ is trivial for any H and G .)

As before, membership in \mathcal{T}_H will (in non-silly cases) at least require that the copies of H cover the edges of $G := G_{n,p}$, but when H is non-Eulerian there is a second requirement: each vertex of G should have odd degree in some copy of H in G (since for any $v \in V(G)$, $\mathcal{W}_H(G)$ will contain graphs in which v has odd degree). For example if H is a pair of triangles joined by a slightly long path and $n^{-1+\varepsilon} < p \ll n^{-2/3}$ for a suitable small ε depending on the length of the path, then (w.h.p.) all edges of G are in copies of H , but most vertices fail to lie in triangles, so have even degree in every copy.

Generalizing \mathcal{Q}_κ , let \mathcal{Q}_H be the class of nonempty graphs G satisfying (i) each edge of G is in a copy of H , and (ii) if H is not Eulerian, then each vertex of G has odd degree in some copy of H ; so we have just said that we “essentially” have $\mathcal{T}_H \subseteq \mathcal{Q}_H$. Though we hesitate to make it a conjecture, we don’t know that the following generalization of Theorem 1.4 is wrong.

Question 1.6. *Could it be that for each (fixed) H ,*

$$\max_p \Pr(G_{n,p} \in \mathcal{Q}_H \setminus \mathcal{T}_H) \rightarrow 0? \quad (7)$$

Understanding when $G_{n,p} \in \mathcal{Q}_H$ w.h.p. is easier, so this would also tell us when \mathcal{T}_H is likely to hold. (Note that in general we don’t expect a statement like Theorem 1.3, since the “threshold” for \mathcal{Q}_H itself may not be sharp.) Even if (7) is not true in general, it seems likely to hold for reasonably nice H (even, say, edge-transitive to start, though this should be much more than is needed). One could also relax (7) to an Erdős-Rényi-like threshold statement; e.g. with $p_{\mathcal{Q}_H} = \min\{p_0 : \Pr(G_{n,p} \in \mathcal{Q}_H) \geq 1/2 \ \forall p \geq p_0\}$,

$$\text{if } p \gg p_{\mathcal{Q}_H} \text{ then } G_{n,p} \in \mathcal{T}_H \text{ w.h.p.}$$

Outline. The rest of the paper is organized as follows. Usage notes conclude the introduction. Section 2 recalls edge space preliminaries and outlines the main points (Lemmas 2.2-2.4) for the proof of Theorem 1.4. Section 3 reviews tools and derives some relatively routine consequences. Section 4 proves Lemma 1.2 and gives the easy derivation of Theorem 1.3 from Theorem 1.4.

The heart of the paper is Sections 5-7, which prove Lemmas 2.4, 2.3 and 2.2. These are, respectively, very easy (modulo a big machine); easy but a little circuitous; and not so easy and *quite* circuitous (and by far the most interesting part of the argument). Finally, Section 8 gives the easy proof of Theorem 1.5, which we postpone as it is unrelated to the rest of the paper.

Usage. Given a graph G , we will use V and E for $V(G)$ and $E(G)$ when the meaning is clear. We will often identify graphs with their edge sets.

For $v \in V$ and $F \subseteq G$ we use $N_F(v) = \{x : vx \in F\}$ and $d_F(v) = |N_F(v)|$. For disjoint $A, B \subseteq V$, $\nabla_F(A, B)$ is the set of F -edges joining A and B , and we use $\nabla_F(A)$ for $\nabla_F(A, V \setminus A)$ —these are the *cuts* of G —and $\nabla_F(v)$ for $\nabla_F(\{v\})$. In all cases we drop the subscripts when $F = G$. As usual $\alpha(G)$ and $\Delta(G)$ (or Δ_G) denote independence number and maximum degree of G . We will sometimes use v_G and e_G for the numbers of vertices and edges of G .

We use $[n]$ for $\{1, \dots, n\}$ (for positive integer n), \log for \ln and $a = (1 \pm b)c$ for $(1 - b)c \leq a \leq (1 + b)c$. Asymptotic notation (\sim , $O(\cdot)$, $\Omega(\cdot)$ and so on), is standard, with $a \ll b$ and $a \asymp b$ replacing $a = o(b)$ and $a = \Theta(b)$ when convenient. Throughout the paper we assume n is large enough to support our various assertions, and usually pretend large numbers are integers.

2 Main points for the Proof of Theorem 1.4

Before outlining the proof of Theorem 1.4, we need to review just a little more background.

2.1 Edge space basics

The edge space $\mathcal{E}(G)$ of a graph G (defined in the paper's second paragraph), being an \mathbb{F}_2 -vector space, comes equipped with a standard inner product: $\langle J, K \rangle = \sum_{e \in E(G)} J(e)K(e) = |J \cap K|$, where the sum and cardinality are interpreted mod 2. (The first expression thinks of J and K as vectors, the second as subgraphs of G .) With this, the orthogonal complement, \mathcal{S}^\perp , of a subspace \mathcal{S} of $\mathcal{E}(G)$ is defined as usual. Then $\mathcal{C}^\perp(G)$, called the *cut space*

of G , consists of the (indicators of) cuts of G (which, note, includes \emptyset); $(\mathcal{C}(G) \cap \mathcal{D}(G))^\perp$ consists of cuts and their complements; and $\mathcal{C}_H^\perp(G)$ is the set of subgraphs of G having even intersection with every copy of H (in G).

As mentioned earlier, $\mathcal{C}_H(G) \subseteq \mathcal{W}_H(G)$ always; dually, $\mathcal{W}_H^\perp(G) \subseteq \mathcal{C}_H^\perp(G)$. In particular, for odd $\kappa \geq 3$,

$$\mathcal{C}^\perp(G) \subseteq \mathcal{C}_\kappa^\perp(G), \text{ and equality here is the same as } G \in \mathcal{T}_\kappa. \quad (8)$$

The next (trivial) observation will be useful at a few points.

Proposition 2.1. *Let G be a graph and $L \subseteq G$, and suppose L', L'' are (respectively) smallest and largest members of the coset $L + \mathcal{C}^\perp(G)$. Then*

$$\forall v \in V \quad d_{L'}(v) \leq d_G(v)/2 \leq d_{L''}(v).$$

(For example if $d_{L'}(v) > d_G(v)/2$, then $L' + \nabla(v) (\in L + \mathcal{C}^\perp(G))$ is smaller than L' .)

In particular, if $G \notin \mathcal{T}_\kappa$, then since $\mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G) \supseteq L + \mathcal{C}^\perp(G)$ for any $L \in \mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G)$, a smallest element F of $\mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G)$ satisfies

$$d_F(v) \leq d_G(v)/2 \quad \forall v \in V. \quad (9)$$

2.2 Structure of the proof

For the rest of the paper we fix an odd $\kappa \geq 5$ (as mentioned earlier, the case $\kappa = 3$ of Theorem 1.4 was proved in [6]), and set $p^* = p_\kappa^*$, $\mathcal{Q} = \mathcal{Q}_\kappa$ and $\mathcal{T} = \mathcal{T}_\kappa$; so our objective, (3), becomes

$$\max_p \Pr(G_{n,p} \in \mathcal{Q} \setminus \mathcal{T}) \rightarrow 0. \quad (10)$$

As sometimes happens, though (10) should become “more true” as p ($> p^*$) grows, some points in the proof run into difficulties for larger p , and it seems easiest to deal first with smaller p and then derive the full statement from this restricted version. The next two lemmas, the first of which is our main point, implement this plan.

Lemma 2.2. *For any fixed K and $p \leq Kp^*$,*

$$\Pr(G_{n,p} \in \mathcal{Q} \setminus \mathcal{T}) \rightarrow 0. \quad (11)$$

(The interest here is really in p at least about p^* , smaller values being handled by Lemma 1.2; see (50).)

Lemma 2.3. *There exists K such that if $p > q := Kp^*$, then*

$$\Pr(G_{n,p} \notin \mathcal{T}) < \Pr(G_{n,q} \notin \mathcal{T}) + o(1).$$

Applying Lemmas 2.3 and 2.2, together with (the 1-statement of) Lemma 1.2 to $p'(n) := \min\{p(n), Kp^*(n)\}$ then easily gives Theorem 1.4. (For n 's with $p(n) > Kp^*$, we have, using Lemma 2.3 for the first inequality and Lemmas 2.2 and 1.2 for the final $o(1)$,

$$\begin{aligned} \Pr(G_{n,p} \in \mathcal{Q} \setminus \mathcal{T}) &< \Pr(G_{n,p'} \notin \mathcal{T}) + o(1) \\ &< \Pr(G_{n,p'} \in \mathcal{Q} \setminus \mathcal{T}) + \Pr(G_{n,p'} \notin \mathcal{Q}) + o(1) = o(1), \end{aligned}$$

and for the remaining n 's we have $p = p'$ and Lemma 2.2 applies directly.)

The following device will play a central role in the proofs of both of these lemmas (so in most of the paper). For the remainder of our discussion we fix some rule that associates with each finite graph G a subgraph $F(G)$ satisfying

$$F(G) = \begin{cases} \emptyset & \text{if } G \in \mathcal{T}, \\ \text{some smallest element of } \mathcal{C}_\kappa^\perp(G) \setminus \mathcal{C}^\perp(G) & \text{if } G \notin \mathcal{T}. \end{cases} \quad (12)$$

We will use this only with $G = G_{n,p}$, so set $F(G_{n,p}) = F$ throughout. A crucial point is that G determines F (for “crucial” see the paragraph preceding Proposition 3.15). That F is a minimizer will be used only to say that it is small and has small degrees, as promised by (9).

With F thus defined we may replace the event $\{G_{n,p} \notin \mathcal{T}\}$ by the more convenient $\{F \neq \emptyset\}$, which in particular allows us to tailor our treatment to the size of a hypothetical F . As we will see, ruling out fairly large F 's is easy—not from scratch, but with the help of a powerful result from [5] (Theorem 3.14 below), which more or less immediately yields:

Lemma 2.4. *For fixed $c > 0$ and $p \gg n^{-(\kappa-2)/(\kappa-1)}$,*

$$\Pr(|F| > cn^2p) \rightarrow 0. \quad (13)$$

Thus the real problem in proving Lemma 2.2, and the most interesting part of the whole business, is dealing with F 's that are small relative to G (but nonempty). Thus far—and a little further; see the preview following the statement of Lemma 7.2—our structure mirrors that of [6], but the (two-page) argument handling this main point there offers no help here.

Remark. In connection with Question 1.6, it seems worth noting here that Lemma 2.4, at least, extends to considerably more general H ; see Section 5 for a little more on this.

3 Tools

3.1 Deviation and correlation

Set

$$\varphi(x) = (1+x)\log(1+x) - x \quad (14)$$

for $x > -1$ and (for continuity) $\varphi(-1) = 1$. We use “Chernoff’s Inequality” in the following form; see for example [13, Thm. 2.1].

Theorem 3.1. *If $X \sim \text{Bin}(n, p)$ and $\mu = \mathbb{E}[X] = np$, then for $t \geq 0$,*

$$\Pr(X \geq \mu + t) \leq \exp[-\mu\varphi(t/\mu)] \leq \exp[-t^2/(2(\mu + t/3))], \quad (15)$$

$$\Pr(X \leq \mu - t) \leq \exp[-\mu\varphi(-t/\mu)] \leq \exp[-t^2/(2\mu)]. \quad (16)$$

For larger deviations the following consequence of the finer bound in (15) will be convenient.

Theorem 3.2. *For $X \sim B(n, p)$ and any K , with $\mu = \mathbb{E}[X] = np$,*

$$\Pr(X > K\mu) < \exp[-K\mu \log(K/e)].$$

(Of course this is only helpful if $K > e$.)

We will make substantial use of the following fundamental lower tail bound of Svante Janson ([12] or [13, Theorem 2.14]), for which we need a little notation. Suppose A_1, \dots, A_m are subsets of the finite set Γ . Let Γ_p be the random subset of Γ gotten by including each $x (\in \Gamma)$ with probability p , these choices made independently. For $j \in [m]$, let I_j be the indicator of the event $\{\Gamma_p \supseteq A_j\}$, and set $X = \sum I_j$, $\mu = \mathbb{E}X = \sum_j \mathbb{E}I_j$ and

$$\overline{\Delta} = \sum \sum \{\mathbb{E}I_i I_j : A_i \cap A_j \neq \emptyset\}. \quad (17)$$

(Note this includes the diagonal terms.)

Theorem 3.3. *With notation as above, for any $t \in [0, \mu]$,*

$$\Pr(X \leq \mu - t) \leq \exp[-\varphi(-t/\mu)\mu^2/\overline{\Delta}] \leq \exp[-t^2/(2\overline{\Delta})].$$

The next result is [13, Lemma 2.46] (originally [12, Lemma 2]).

Lemma 3.4. *For events A_1, \dots, A_n in a probability space, and $\mu = \sum \Pr(A_i)$,*

$$\begin{aligned} \Pr(\text{some } \mu + t \text{ independent } A_i \text{'s occur}) &\leq \exp[-\mu\varphi(t/\mu)] \\ &\leq \exp[-t^2/(2(\mu + t/3))]. \end{aligned}$$

Note the bound here is the same as the one in (15), which is thus contained in Lemma 3.4. (Strictly speaking, [12] and [13] state Lemma 3.4 only in setting of Theorem 3.3, but the proofs there are valid for the version here.) Lemma 3.4 implies the weaker but sometimes convenient

$$\Pr(\text{some } l \text{ independent } A_i\text{'s occur}) \leq \mu^l/l! \leq (e\mu/l)^l \quad (18)$$

observed in [9] (or see [2, Lemma 8.4.1]).

The setting for the next theorem is a finite product probability space $\Omega = \prod_{i=1}^t \Omega_i$ with each factor linearly ordered. As usual an event $A \subseteq \Omega$ is *increasing* if its indicator is a nondecreasing function (with respect to the product order on Ω) and *decreasing* if its complement is increasing. The seminal ‘‘correlation inequality’’ is essentially due to Harris [11]:

Theorem 3.5. *If $A, B \subseteq \Omega$ are either both increasing or both decreasing, then*

$$\Pr(A \cap B) \geq \Pr(A) \Pr(B);$$

if one is increasing and the other decreasing then the inequality is reversed.

3.2 Density generics

From now on we use G for $G_{n,p}$ and V for $[n] = V(G)$. Theorems 3.1 and 3.2 easily imply the next two standardish propositions, whose proofs we omit.

Proposition 3.6. *For $p \gg n^{-1} \log n$, w.h.p.*

$$|G| \sim n^2 p/2 \quad \text{and} \quad d(v) \sim np \quad \forall v \in V.$$

(Of course the second conclusion implies the first, which just needs $p \gg n^{-2}$.)

Proposition 3.7. (a) *For any $\varepsilon > 0$ there is a K such that w.h.p. for all disjoint $S, T \subseteq V$ with $|S|, |T| > Kp^{-1} \log n$*

$$|\nabla_G(S, T)| = (1 \pm \varepsilon)|S||T|p$$

and

$$|G[S]| = (1 \pm \varepsilon) \binom{|S|}{2} p.$$

(b) *For $K > 3$ w.h.p.*

$$|G[S]| < K|S| \log n \quad \text{for all } S \subseteq V \text{ with } |S| \leq Kp^{-1} \log n.$$

(c) *For each $\varepsilon > 0$ there is a K such that if $p > Kn^{-1} \log n$ then w.h.p.*

$$|\nabla_G(S)| = (1 \pm \varepsilon)|S|(n - |S|) \quad \forall S \subseteq V.$$

Proposition 3.8. *For fixed $\varepsilon > 0$ and $p \gg 1/n$, w.h.p.: if $H \subseteq G$ satisfies*

$$d_H(v) > (1 - \varepsilon)np/2 \quad \forall v \in V, \quad (19)$$

then no component of H has size less than $(1 - 2\varepsilon)n/2$.

Proof. For a given $W \subseteq V$ of size $w < (1 - 2\varepsilon)n/2$, let $\chi = |G[W]|$. Then $\mu := \mathbb{E}\chi = \binom{w}{2}p < w^2p/2$, while if W is a component of an H satisfying (19) then

$$\chi \geq |H[W]| > w(1 - \varepsilon)np/4 > \frac{(1 - \varepsilon)n}{2w}\mu =: K\mu.$$

But (since $K > (1 - \varepsilon)/(1 - 2\varepsilon) = 1 + \Omega(1)$) Theorems 3.1 and 3.2 give

$$\gamma_w := \Pr(\chi > K\mu) < \begin{cases} \exp[-\Omega(\mu)] & \text{if } K < e^2 \text{ (say),} \\ \exp[-K\mu \log(K/e)] & \text{otherwise.} \end{cases}$$

Thus, with sums over $w \in (0, (1 - 2\varepsilon)n/2)$, the probability that some H as in the lemma admits a component of size less than $(1 - 2\varepsilon)n/2$ is less than

$$\sum \binom{n}{w} \gamma_w < \sum \exp[w \log(en/w)] \gamma_w,$$

which for $p \gg 1/n$ is easily seen to be $o(1)$. \square

Finally, we need to know a little about the adjacency matrix, $A(G)$, of G . A version of (20) below was proved in [10] (see also [1]) and (21) is shown (e.g.) in [17].

Proposition 3.9. *Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of $A(G)$ and v_1, v_2, \dots, v_n associated orthonormal eigenvectors, say with $\max_j v_{1,j} > 0$. If $p \gg n^{-1} \log n$, then w.h.p.*

$$\lambda_1 \sim np \quad \text{and} \quad \max\{\lambda_2, |\lambda_n|\} < (2 + o(1))\sqrt{np}. \quad (20)$$

If $p > n^{-1} \log^6 n$, then w.h.p.

$$\max_j v_{1,j} < (1 + o(1)) \min_j v_{1,j}. \quad (21)$$

3.3 Path counts

This section discusses what can be said about the numbers of paths of various lengths joining pairs of vertices in a random graph.

Notation. For $l \geq 1$ and (distinct) $x, y \in V$, we use $P^l(x, y)$ for the set of P_l 's (l -edge paths) in G joining x and y , $\tau^l(x, y)$ for $|P^l(x, y)|$, and $\sigma^l(x, y)$

for the maximum size of a collection of internally disjoint P_l 's of G joining x and y . (Though $l = 1$ is uninteresting, it's convenient to allow this.)

In this section we use $V(P)$ for the set of *internal* vertices of a path P and write $\Gamma_{x,y}^l$ for the graph on $P^l(x,y)$ with $P \sim Q$ iff $V(P) \cap V(Q) \neq \emptyset$.

Conveniently, most of what we need here has been worked out (in far greater generality) by Joel Spencer in [22] (see also [2, Section 8.5]), and we begin with two special cases of what's proved there.

Theorem 3.10. *For any $l \geq 2$ and $\varepsilon > 0$ there exists K such that if $n^{l-1}p^l \geq K \log n$, then w.h.p.*

$$\tau^l(x,y) = (1 \pm \varepsilon)n^{l-1}p^l \quad \forall \{x,y\} \in \binom{V}{2}. \quad (22)$$

Proposition 3.11. *For any $l \geq 1$ and $\delta > 0$, if $n^{2l-3}p^{2l-1} < n^{-\delta}$ then w.h.p.*

$$\tau^i(x,y) - \sigma^i(x,y) < C \quad \forall \{x,y\} \in \binom{V}{2}, \quad i \in [l], \quad (23)$$

where C depends only on l and δ .

We note for use below that the assumption on p in Proposition 3.11 implies

$$n^{l-2}p^{l-1} < n^{-\zeta}, \quad (24)$$

with $\zeta = (1 + \delta(l-1))/(2l-1)$ ($= \Omega(1)$). Strictly speaking, the proposition is a little stronger than what one gets from [22], where the assumption would be $n^{l-1}p^l = O(\log n)$. (The $n^{2l-3}p^{2l-1}$ is more or less the expected number of non-edge-disjoint pairs of paths joining a given x and y .)

Proposition 3.11, though not difficult, is a key point in Spencer's proof of Theorem 3.10, and from our perspective is in a sense the main point, since, as indicated in the remark below, it easily gives the latter when combined with Lemma 3.4. Since the proof of the proposition itself is not so easy to extract from Spencer's presentation (see his "third part" on p. 253), we next sketch an argument along lines similar to his for the present situation.

Proof of Proposition 3.11. It is enough to handle $i = l$ (since the assumption on p implies a stronger assumption when we replace l by $i < l$). Noting that $\tau^l(x,y) - \sigma^l(x,y) \leq |E(\Gamma_{x,y}^l)|$, we find that (23) (with an appropriate C) holds at x,y provided

- (i) the maximum number of vertices in a component of $\Gamma_{x,y}^l$ is $O(1)$ and

(ii) the maximum size of an induced matching in $\Gamma_{x,y}^l$ is $O(1)$;

so we want to say that w.h.p. these conditions hold for all x, y . (Of course replacing (i) by an $O(1)$ bound on degrees would also suffice.)

For (i) we show that, for some fixed M , w.h.p. there do not exist x, y and a collection, Q_1, \dots, Q_M , of P_l 's joining x and y such that, for $i \geq 2$, $V(Q_i)$ meets, but is not contained in, $\cup_{j < i} V(Q_j)$. This bounds (by $(l-2)M+1$) the number of internal vertices (of G) in the paths belonging to a component of $\Gamma_{x,y}^l$, so gives (i).

Suppose Q_1, \dots, Q_M are P_l 's joining x and y , with $R_i = \cup_{j \leq i} Q_j$ and, for $i \geq 2$, $|E(Q_i) \setminus E(R_{i-1})| = b_i$ and $|V(Q_i) \setminus V(R_{i-1})| = a_i \in [1, l-2]$. Then $b_i \geq a_i + 1$ and $a_i \leq l-2$ imply $n^{a_i} p^{b_i} \leq n^{l-2} p^{l-1}$ (for $i \geq 2$) and

$$n^{|V(R_M)|} p^{|E(R_M)|} \leq np(n^{l-2} p^{l-1})^M, \quad (25)$$

which is thus an upper bound on the probability of finding, for a given x, y , (Q_1, \dots, Q_M) as above of a given isomorphism type (defined in the obvious way). So the probability that there are such Q_i 's for *some* x, y (and some isomorphism type) is $O(n^3 p(n^{l-2} p^{l-1})^M) = O(n^3 p n^{-\zeta M})$ (see (24)), so is $o(1)$ for large enough M .

The argument for (ii) is similar. Here we want to rule out, again for some fixed M , existence of P_l 's, say $Q_1, R_1, \dots, Q_M, R_M$, joining some specified x, y , with $V(Q_i) \cap V(R_i) \neq \emptyset$ and the $V(Q_i)$'s and $V(R_i)$'s otherwise disjoint. A discussion like the one above shows that for any such sequence, with $|\cup_i (E(Q_i) \cup (E(R_i)))| = b$ and $|\cup_i (V(Q_i) \cup (V(R_i)))| = a$, we have

$$n^a p^b < (n^{2l-3} p^{2l-1})^M < n^{-M\delta}, \quad (26)$$

which bounds the probability of existence by $O(n^{2-M\delta})$. \square

Remark. The lower bound in Theorem 3.10 is given by Theorem 3.3 (a recent development at the time). The main issue for the upper bound is handling p with $n^{l-1} p^l \asymp \log n$, for which Proposition 3.11 allows replacing τ^l by σ^l . This is then naturally handled by Lemma 3.4, replacing Spencer's nice, if slightly *ad hoc* approach based on maximal disjoint families.

Theorem 3.10 and Proposition 3.11 (with bits of Section 3.1) easily imply the following bounds on the $\tau^l(x, y)$'s.

Corollary 3.12. *W.h.p. for all (distinct) vertices x, y ,*

$$\tau^l(x, y) \sim n^{l-1} p^l \quad \text{if } n^{l-1} p^l = \omega(\log n), \quad (27)$$

$$\tau^l(x, y) = O(\log n) \quad \text{if } n^{l-1} p^l = O(\log n), \quad (28)$$

$$\tau^l(x, y) = O(1) \quad \text{if } n^{l-1} p^l < n^{-\Omega(1)}. \quad (29)$$

Proof. The first two items are easy consequences of Theorem 3.10: (27) is immediate and (28) is given by the observation that, for K as in the theorem (for some specified ε) and p_0 defined by $n^{l-1}p_0^l = K \log n$, the theorem implies that w.h.p.

$$\tau^l(x, y) \leq (1 + \varepsilon)n^{l-1}(\max\{p, p_0\})^l \quad \forall \{x, y\} \in \binom{V}{2} \quad (30)$$

(since the probability of the event in (30) decreases as p increases below p_0).

For (29), suppose $n^{l-1}p^l < n^{-\alpha}$, with $\alpha > 0$ fixed. Since this implies $n^{2l-3}p^{2l-1} < n^{-\delta}$ with $\delta = \delta_\alpha > 0$ fixed, Proposition 3.11 says it suffices to show that for given x, y and suitable fixed D (depending on α),

$$\Pr(\sigma^l(x, y) > D) = o(n^{-2}).$$

But (18) bounds this probability by $\exp[-D \log(n^\alpha D/e)]$, which is $o(n^{-2})$ for large enough D . \square

We will also sometimes need *lower* bounds on path counts, as summarized in the next result, which again follows easily from what we already know.

Corollary 3.13. *For any $l \geq 2$ there is a K such that if $n^{l-1}p^l \geq K \log n$, then w.h.p. $\sigma^l(x, y) = \Omega(\pi)$ for all x, y , with $\pi = \pi(n, p)$ equal to*

$$n^{l-1}p^l \quad \text{if } n^{l-2}p^{l-1} < n^{-\Omega(1)}, \quad (31)$$

$$n^{l-1}p^l / \log n \quad \text{if } n^{-o(1)} < n^{l-2}p^{l-1} = O(\log n), \quad (32)$$

$$np \quad \text{if } n^{l-2}p^{l-1} = \omega(\log n). \quad (33)$$

(Of course in view of Proposition 3.6, $(1 + o(1))np$ is a trivial upper bound.)

Proof. Let K be as in Theorem 3.10, for the given l and, say, $\varepsilon = 1/2$ (since we don't worry about constants). Since the theorem says that w.h.p. $|V(\Gamma_{x,y}^l)| > \Omega(n^{l-1}p^l)$ for all x, y , the present assertion(s) will follow if we show

$$\text{w.h.p. } \Delta(\Gamma_{x,y}^l) = O(n^{l-1}p^l/\pi) \quad \forall x, y, \quad (34)$$

where we use the the trivial $\alpha \geq |V|/\Delta$ (recall Δ and α are maximum degree and independence number and note $\sigma^l(x, y) = \alpha(\Gamma_{x,y}^l)$). Now the degree in $\Gamma_{x,y}^l$ of a given vertex Q (that is, a P_l joining x and y) is at most

$$\sum_v \sum_i \tau^i(x, v) \tau^{l-i}(v, y) \leq (l-1)^2 \max\{\tau^i(x, v) \tau^{l-i}(v, y)\}, \quad (35)$$

where the sums are over $v \in V(Q)$ and $i \in [l-1]$, and the max is over $i \in [l-1]$ and $v \in V \setminus \{x, y\}$ (the initial $(l-1)^2$ is of course irrelevant). On the other hand, Corollary 3.12 (with i in place of l) says that w.h.p. we have, for all u, v :

$\tau^i(u, v) < O(1)$ if either $i \leq l - 2$ and p is as in (31) or (32), or $i = l - 1$ and p is as in (31),

and $\tau^i(u, v) < O(\max\{n^{i-1}p^i, \log n\})$ in general; and combining these bounds with (35) easily yields (34). \square

Finally, in connection with the setup introduced at (12), we will need the following simple observation:

$$xy \in F \implies |F| \geq \sigma^{\kappa-1}(x, y) + 1. \quad (36)$$

Proof. Since F lies in $\mathcal{C}_\kappa^\perp(G)$, it must contain a second edge of each κ -gon of G containing xy , and there is a set of $\sigma^{\kappa-1}(x, y)$ such κ -gons that share no edges except xy . \square

3.4 Stability

The following statement is an instance of a major result of Conlon and Gowers [5]. As mentioned in Section 2.2, this is the main (essentially only) ingredient in the proof of Lemma 2.4 given in Section 5.

Theorem 3.14. *For each odd $\kappa \geq 3$ and $\varepsilon > 0$ there is a C such that if $p > Cn^{-(\kappa-2)/(\kappa-1)}$, then w.h.p. every C_κ -free subgraph of $G = G_{n,p}$ of size at least $|G|/2$ can be made bipartite by deleting at most $\varepsilon n^2 p$ edges.*

This (or the more general result of [5]) is a “sparse random” analogue of the Erdős-Simonovits “Stability Theorem” [8, 21] that was conjectured by Kohayakawa *et al.* in the seminal [16].

As mentioned in Section 2, Lemma 2.4 can be considerably extended; in fact we can prove something similar with C_κ replaced by a general H , though not always with the lower bound on p that would correspond to a positive answer to Question 1.6. See Section 5 for a precise statement.

3.5 Coupling

A central role in the proofs of Lemmas 2.2 and 2.3 is played by the usual coupling of $G := G_{n,p}$ and $G_0 := G_{n,q}$, where p will always be the value we’re really interested in and $q < p$ will depend on what we’re trying to do. A standard description:

Let $\lambda_e, e \in E(K_n)$, be chosen uniformly and independently from $[0, 1]$ and set

$$G = \{e : \lambda_e < p\}, \quad G_0 = \{e : \lambda_e < q\}.$$

In particular $G_0 \subseteq G$. Probabilities in the proofs of Lemmas 2.2 and 2.3 refer to the joint distribution of G and G_0 .

We will get most of our leverage from two alternate ways of viewing the choice of the pair (G, G_0) :

- (A) Choose G first; thus we choose $G (= G_{n,p})$ in the usual way and let G_0 be the (“ (q/p) -random”) subset of G gotten by retaining edges of G with probability q/p , these choices made independently (a.k.a. *percolation on G*).
- (B) Choose G_0 first; that is, we choose $G_0 (= G_{n,q})$ in the usual way, define p' by $(1 - q)(1 - p') = 1 - p$, and let G be the random superset of G_0 gotten by adding each edge of $\overline{G_0}$ to G_0 with probability p' , these choices again made independently.

We will often refer to these as “coupling down” and “coupling up” (respectively).

The proof of Lemma 2.3 is based naturally (or inevitably) on the viewpoint in (A); namely, we show that (with p, q as in the lemma) if $G = G_{n,p}$ is “bad” (meaning $G \notin \mathcal{T}$) then the coupled $G_0 = G_{n,q}$ is likely to be bad as well. For the proof of Lemma 2.2, viewpoint (B) is the primary mover, though the role of (A) is also crucial.

With reference to the setup introduced at (12), when working with $G = G_{n,p}$ and $G_0 = G_{n,q}$ as above, we set $F_0 = G_0 \cap F$ (a (q/p) -random subset of F ; note this has nothing to do with $F(G_0)$, which will play no role here). Then automatically

$$F_0 \in \mathcal{C}_\kappa^\perp(G_0), \quad (37)$$

since $F_0 \cap C = F \cap C$ for any κ -gon C of G_0 .

We will want to say that certain features of (G, F) are reflected in (G_0, F_0) . A simple but crucial point here is that there is no summing (of probabilities) over possible F ’s, since there is just one F for each G . The following proposition will be sufficient for our purposes.

Proposition 3.15. *With the above setup, for any p, q and $g = g(n) = \omega(1)$, w.h.p.*

$$|F_0| \sim |F|q/p \quad \text{if } |F| > gp/q$$

and

$$d_{F_0}(v) \begin{cases} \sim d_F(v)q/p & \forall v \text{ with } d_F(v) > (g \log n)p/q, \\ < 3g \log n & \forall v \text{ with } d_F(v) \leq (g \log n)p/q. \end{cases}$$

(This is true for any rule that specifies a particular subgraph (in place of F) for each graph; but we will only use it with F ($= F(G)$), so just give the statement for this case.)

Proof. These are straightforward applications of Theorems 3.1 and 3.2, so we will be brief. For the first assertion we want to say that for any fixed $\varepsilon > 0$,

$$\Pr(|F| > gp/q \wedge (|F_0| \neq (1 \pm \varepsilon)|F|q/p)) \rightarrow 0.$$

But the probability here is less than

$$\Pr(|F_0| \neq (1 \pm \varepsilon)|F|q/p \mid |F| > gp/q),$$

which by Theorem 3.1 is less than $\exp[-\Omega(\varepsilon^2 g)]$.

The second assertion (pair of assertions) is similar, following from

$$\begin{aligned} \sum_v \Pr(d_{F_0}(v) \neq (1 \pm \varepsilon)d_F(v) \mid d_F(v) > (g \log n)p/q) &< n \exp[-\Omega(\varepsilon^2 g \log n)] \\ &= o(1) \end{aligned}$$

for any fixed $\varepsilon > 0$, and (now switching to Theorem 3.2)

$$\begin{aligned} \sum_v \Pr(d_{F_0}(v) > 3g \log n \mid d_F(v) \leq (g \log n)p/q) &< n \exp[-(3g \log n) \log(3/e)] \\ &= o(1). \end{aligned} \quad \square$$

4 Two simple points

Here we dispose of Lemma 1.2 and the derivation of Theorem 1.3 from Theorem 1.4. (Recall we are using G for $G_{n,p}$ and V for $V(G)$.)

Proof of Lemma 1.2. We begin with the 1-statement, a typical application of Theorem 3.3. We assume $p > (1 + \varepsilon)p^*$ and $p = O(p^*)$ (as we may, since for larger p , the 1-statement is contained in Theorem 3.10). Given $x, y \in V$, let the A_i 's (in the paragraph preceding Theorem 3.3) be the edge sets of the $(\kappa - 1)$ -paths joining x and y in K_n ; so $X = \tau^{\kappa-1}(x, y)$, $\mu \sim n^{\kappa-2}p^{\kappa-1}$ and $\overline{\Delta} = \mu + O(\mu n^{\kappa-3}p^{\kappa-2}) \sim \mu$. Thus (note $\varphi(-1) = 1$) Theorem 3.3 gives

$$\Pr(\tau^{\kappa-1}(x, y) = 0) \leq \exp[-(1 - o(1))\mu]. \quad (38)$$

So the probability that \mathcal{Q} ($= \mathcal{Q}_\kappa$) fails—that is, that there is some xy in G with $\tau^{\kappa-1}(x, y) = 0$ —is less than

$$\binom{n}{2} p e^{-(1-o(1))\mu} < \exp[\log(n^2 p) - (1 - o(1))\mu] = o(1)$$

(since $\mu > (1 - o(1))(1 + \varepsilon)^{\kappa-1}(\kappa/(\kappa - 1)) \log n \sim (1 + \varepsilon)^{\kappa-1} \log(n^2 p)$).

For the 0-statement we use the second moment method (see e.g. Chapter 4 of [2]) and, again, Theorem 3.3. Let Z_{xy} be the indicator of the event $\{xy \in G\} \wedge \{\tau^{\kappa-1}(x, y) = 0\}$ ($x, y \in V$) and $Z = \sum Z_{xy}$. Theorem 3.5 gives $\Pr(\tau^{\kappa-1}(x, y) = 0) > (1 - p^{\kappa-1})^{n^{\kappa-2}} > \exp[-\mu - o(1)]$ (μ as above), whence

$$\mathbb{E}[Z_{xy}] > p \exp[-\mu - o(1)]. \quad (39)$$

In particular $\mathbb{E}[Z] = \omega(1)$ (using $p < (1 - \varepsilon)p^*$ and ignoring the rather trivial case $p = O(n^{-2})$), so for $\mathbb{E}Z^2 \sim \mathbb{E}[Z]^2$ (which gives the 0-statement *via* Chebyshev's Inequality), it's enough to show

$$\mathbb{E}[Z_{xy}Z_{uv}] < (1 + o(1))\mathbb{E}[Z_{xy}]^2$$

for distinct $\{x, y\}, \{u, v\} \in \binom{V}{2}$, which in view of (39) follows from

$$\begin{aligned} \mathbb{E}[Z_{xy}Z_{uv}] &\leq p^2 \Pr(\tau^{\kappa-1}(x, y) = \tau^{\kappa-1}(u, v) = 0) \\ &\leq p^2 \exp[-(1 - O(n^{\kappa-3}p^{\kappa-2}))2\mu] = p^2 \exp[-2\mu + o(1)]. \end{aligned}$$

Here the first inequality is given by Theorem 3.5 (since the events $\{xy, uv \in G\}$ and $\{\tau^{\kappa-1}(x, y) = \tau^{\kappa-1}(u, v) = 0\}$ are increasing and decreasing respectively), and the second by Theorem 3.3, where the A_i 's are the $(\kappa - 1)$ -edge paths joining either x and y or u and v , for which $\mathbb{E}X \sim 2\mu$ (recall X is the number of A_i 's that occur) and it's easy to see that $\overline{\Delta} - \mu = O(n^{2\kappa-5}p^{2\kappa-3}) = O(n^{\kappa-3}p^{\kappa-2})\mu$ ($= o(\mu)$). \square

Proof that Theorem 1.4 implies Theorem 1.3. This is again routine and we aim to be brief. Lemma 1.2 gives the 1-statement (which is the interesting part). For the 0-statement, it is enough to say that for p in the stated range, $G = G_{n,p}$ w.h.p. contains an edge lying in a cycle but not in a C_κ . This is again given by Lemma 1.2 if p is large enough that *all* edges are in cycles (w.h.p), which is true if $p > (1 + \Omega(1)) \log n/n$ (again, see [13, p. 105]). For smaller p , w.h.p. G contains cycles of length $\omega(1)$ if $p > (1 - o(1))/n$ and of length (say) $\Omega(n^3)$ if $p > (1 + \Omega(1))/n$ (see e.g. [13, Thm. 5.18(i)]). On the other hand, since the expected number of C_κ 's in G is less than $(np)^\kappa$, the number of edges in C_κ 's is w.h.p. less than $\omega \cdot (np)^\kappa$ for any $\omega = \omega(1)$; so in the range under discussion, the C_κ 's w.h.p. don't cover the edges of even one longest cycle in G . \square

5 Proof of Lemma 2.4

Here we give the easy proof of Lemma 2.4 and then state the extension to general H mentioned in the remark following the lemma.

For the lemma it's enough to show that the conclusions of Proposition 3.6, Theorem 3.14 and Proposition 3.7(c), the latter two with $\varepsilon = c/3$, imply $|F| < cnp^2$ (deterministically).

Let F' be a largest element of $F + \mathcal{C}^\perp(G)$. Then $|F'| \geq |G|/2$ (by Proposition 2.1), so, since F' is \mathcal{C}_κ -free, the conclusion of Theorem 3.14 gives an $A \subseteq V$ with

$$|F' \setminus \nabla_G(A)| < \varepsilon n^2 p. \quad (40)$$

To finish we just check that (under our assumptions), (40) implies

$$(|F| \leq) \quad |F' \triangle \nabla_G(A)| < 3\varepsilon n^2 p :$$

the conclusion of Proposition 3.7(c) gives $|\nabla_G(A)| < (1 + \varepsilon)n^2 p/4$, whence

$$|\nabla_G(A) \setminus F'| \leq (1 + \varepsilon)n^2 p/4 - (|G|/2 - \varepsilon n^2 p) < 2\varepsilon n^2 p$$

(where we again used Proposition 3.6 to say $|G| \sim n^2 p/2$). \square

Generalization. (We continue to use G for $G_{n,p}$.) For this discussion we restrict to H with $e_H \geq 2$ (so $v_H \geq 3$). For such an H , set

$$m_2(H) = \max \left\{ \frac{e_K - 1}{v_K - 2} : K \subseteq H, v_K \geq 3 \right\}. \quad (41)$$

This parameter plays a central role in various contexts, in particular in results more or less related to (the general version of) Theorem 3.14; see e.g. [19] for an overview.

Theorem 5.1. *For any fixed H the following is true. For any $\varepsilon > 0$ there is a C such that if $p > Cn^{-1/m_2(H)}$ then w.h.p.: for each $F \in \mathcal{C}_H^\perp(G)$ there is an $X \in \mathcal{W}_H^\perp(G)$ with $|F \triangle X| < \varepsilon n^2 p$; in particular, if $\mathcal{C}_H(G) \neq \mathcal{W}_H(G)$, then*

$$\min\{|F| : F \in \mathcal{C}_H^\perp(G) \setminus \mathcal{W}_H^\perp(G)\} < \varepsilon n^2 p.$$

Since we aren't using this (and since the present work is already too long), we refer to [4, Sec. 4.8] for the proof, here just mentioning that the main ingredients are the “container” machinery of [3, 20] and the following analogue of the Erdős-Simonovits “Stability Theorem” [8, 21]. (The role of this lemma in the proof of Theorem 5.1 is similar to that of Erdős-Simonovits in the proofs of Theorem 3.14 in [3, 20].)

For any H and $F \subseteq E(K_n)$, let $\tau_H(F)$ be the number of copies of H in K_n (say unlabelled) having odd intersection with F .

Lemma 5.2. *For any fixed graph H and $\varepsilon > 0$, there is a $\delta > 0$ such that if $F \subseteq E(K_n)$ satisfies $\tau_H(F) < \delta n^{v_H}$, then there is an $X \in \mathcal{W}_H^\perp(K_n)$ with $|F \Delta X| < \varepsilon n^2$.*

Remarks. Notice that Theorem 5.1 *contains* an extension of Lemma 2.4, whereas in the preceding discussion we did need a few lines to get from Theorem 3.14 to the lemma. But the two theorems live in somewhat different worlds, since Theorem 3.14 assumes only that F is C_κ -free, which is much weaker than requiring that it have even intersection with every C_κ .

As mentioned in Section 3.4, the value $n^{-1/m_2(H)}$ is not necessarily what's needed for Question 1.6. For instance, if H is two triangles joined by a P_1 , then $m_2(H) = 2$ (take K to be one of the triangles), but the range where the question is most interesting (the point at which \mathcal{Q}_H becomes likely) is $p \asymp n^{-2/3} \log^{1/3} n$, corresponding to all *vertices* being in triangles. On the other hand, in natural cases—e.g. the (“balanced”) H 's for which $K = H$ achieves the max in (41)—Theorem 5.1 does give what should be the correct extension of Lemma 2.4. (It would be interesting to see if one could push the theorem to give the correct extension in general; with our current approach this would mainly require a fairly significant extension of what we are getting from “containers,” and we haven't yet thought about plausibility.)

6 Proof of Lemma 2.3

By Corollary 3.13 with $l = \kappa - 1$, there is a $K > 1$ such that if $p > Kp^*$, then w.h.p.

$$\text{every } \{x, y\} \in \binom{V}{2} \text{ satisfies } \sigma^{\kappa-1}(x, y) = \Omega(\pi) \quad (42)$$

(where $\pi = \pi(n, p)$ is as in the corollary). We work in the coupling framework of Section 3.5, taking $q = Kp^*$ and $G_0 = G_{n,q}$.

For Lemma 2.3 it is of course enough to show

$$\Pr(\{G \notin \mathcal{T}\} \wedge \{G_0 \in \mathcal{T}\}) \rightarrow 0. \quad (43)$$

Note that $G_0 \in \mathcal{T}$ implies $F_0 \in \mathcal{C}^\perp(G_0)$, since we always have $F_0 \in \mathcal{C}_\kappa^\perp(G_0)$ (see (37)); thus (43) will follow from

$$\Pr(\{F \neq \emptyset\} \wedge \{F_0 \in \mathcal{C}^\perp(G_0)\}) \rightarrow 0. \quad (44)$$

So it will be enough to show that

$$F_0 \notin \mathcal{C}^\perp(G_0) \quad (45)$$

follows (deterministically) from

$$F \neq \emptyset \quad (46)$$

combined with various statements that we already know hold w.h.p. This is not hard, but is more circuitous than one might wish. Roughly we show that, barring occurrence of some low probability event, (i) presence of even one edge in F forces F to be large enough (not very large) that $F_0 \neq \emptyset$, and (ii) F_0 is not substantial enough to meet all xy -paths in $G_0 - xy$ for an $xy \in F_0$, so any such xy is contained in a cycle witnessing (45).

A convention. To slightly streamline the presentation we agree that in this argument, appeals to a probabilistic statement X —e.g. “ X implies” or “by X ”—actually refer to *the conclusion of X* , which conclusion will always be something that X says holds w.h.p. See the references to (42), Lemma 2.4 and Proposition 3.15 in the next paragraph for first instances of this.

If (46) holds, then (42) and (36) (for the lower bound) together with Lemma 2.4 (for the upper) imply that

$$\Omega(\pi) < |F| < n^2 p / 10. \quad (47)$$

Since $\pi q / p \gg 1$, the lower bound in (47) and the first part of Proposition 3.15 give $|F_0| \sim |F|q/p$, so

$$0 \neq |F_0| < (1 + o(1))n^2 q / 10. \quad (48)$$

In addition, Proposition 3.6, (9) and the second part of Proposition 3.15 give

$$d_{F_0}(v) < (1 + o(1))nq/2 \quad \forall v \in V.$$

Thus, setting $H_0 = G_0 \setminus F_0$ and recalling the approximate (nq) -regularity of G_0 given by Proposition 3.6, we have

$$d_{H_0}(v) > (1 - o(1))nq/2 \quad \forall v \in V. \quad (49)$$

Now choose an $xy \in F_0$ (recall (48) says $F_0 \neq \emptyset$) and let X, Y be the H_0 -components of x and y . By (49) and Proposition 3.8 (applied to G_0), we have $|X|, |Y| > n/3$, which implies $X = Y$: otherwise X and Y are disjoint and we have the contradiction

$$(1 - o(1))n^2 q / 9 < |\nabla_{G_0}(X, Y)| \leq |F_0| < (1 + o(1))n^2 q / 10,$$

where the first inequality is given by Proposition 3.7(a) (applied to G_0), the second holds because $\nabla_{G_0}(X, Y) \subseteq F_0$, and the third is given by (48).

But this (i.e. $X = Y$) gives an xy -path in H_0 , and adding xy to this path produces a cycle meeting F_0 only in xy ; so we have (45).

7 Proof of Lemma 2.2

Here we first introduce the main assertions, Lemmas 7.1 and 7.2, underlying Lemma 2.2, and prove the latter assuming them. The supporting lemmas are then proved in Sections 7.1 and 7.2.

Note that for the proof of Lemma 2.2, Lemma 1.2 allows us to restrict attention to the range

$$(1 - \varepsilon)p^* < p < Kp^* \quad (50)$$

(for any fixed $\varepsilon > 0$), and that Lemma 2.4 says it's enough to show that for a given $\lambda = \lambda(n) \rightarrow 0$,

$$\Pr(\{G_{n,p} \in \mathcal{Q}\} \wedge \{0 < |F| < \lambda n^2 p\}) \rightarrow 0. \quad (51)$$

We again work with the coupling of Section 3.5, now taking $q = \vartheta p$ with a *fixed* $\vartheta \in (0, 1)$ small enough to support the discussion below (the rather mild constraints on ϑ are at (62) and (69)). Define the random variables α and α_0 by

$$|F| = \alpha n^2 p / 2 \quad \text{and} \quad |F_0| = \alpha_0 n^2 q / 2. \quad (52)$$

Definitions. Henceforth a *path* (with length unspecified) is a $P_{\kappa-1}$ (and an xy -path is a path whose endpoints are x and y). Our paths will always lie in G and often in G_0 . We now write $\sigma(x, y)$ for $\sigma^{\kappa-1}(x, y)$ (recall from Section 3.3 that this is the maximum size of a set of internally disjoint xy -paths in G), and $\sigma_0(x, y)$ for the analogous quantity in G_0 . For $S \subseteq G$, a path P is *S-central* if it contains an odd number of edges of S , at least one of which is internal. Let $\sigma(x, y; S)$ be the maximum size of a collection of internally disjoint S -central xy -paths, and $\sigma_0(x, y; S)$ the corresponding quantity in G_0 . An (S, t) -rope is a P_t whose terminal edges lie in S . Set

$$R(S) = \{\{x, y\} \in \binom{V}{2} : \sigma_0(x, y; S) > .25 n^{\kappa-2} q^{\kappa-1}\} \quad (53)$$

and define events

$$\mathcal{R} = \{|F \cap R(F_0)| \geq .12 \alpha n^2 p\}$$

and

$$\mathcal{P} = \{0 < |F| < \lambda n^2 p\}$$

(the second conjunct in (51)).

Lemma 7.1. *There is a fixed $\varepsilon > 0$ such that for p as in (50), w.h.p.*

$$G \in \mathcal{Q} \wedge \mathcal{P} \Rightarrow G \in \mathcal{R}. \quad (54)$$

(In other words, $\Pr(G \in \mathcal{Q} \wedge \mathcal{P} \wedge \overline{\mathcal{R}}) \rightarrow 0$. Of course \mathcal{R} holds trivially if $F = \emptyset$, so it's only the upper bound in \mathcal{P} that's of interest here.)

Remarks. For $\{x, y\} \in \binom{V}{2}$, $\sigma_0(x, y)$ should be around $n^{\kappa-2} q^{\kappa-1}$. Lemma 7.1 says that, provided $G \in \mathcal{Q} \wedge \mathcal{P}$, it's likely that for a decent fraction of the edges xy of F , even $\sigma_0(x, y, F_0)$ is of this order of magnitude—which is *unnatural* if F_0 is small relative to G_0 (since then paths should typically avoid F_0). Viewed from Lemma 7.1 the parity requirement in the definition of “central” may look superfluous, since a path of G_0 joining ends of an edge of F necessarily has odd intersection with F_0 ; but this extra condition will later play a brief but important role in justifying (58).

For the next lemma we temporarily expand the range of q and G_0 , assuming only what's needed for the proof (though we will use the lemma only with q and G_0 as above).

Lemma 7.2. *For fixed $t \geq 3$, $q = q(n) > n^{-1} \log^6 n$ and $G_0 = G_{n,q}$, w.h.p.: for $S \subseteq G_0$, say with $|S| = \beta n^2 q/2$, the number of (S, t) -ropes in G_0 is*

$$O(\max\{\beta^2 n^{t+1} q^t, \beta n^{t/2+2} q^{t/2+1}\}). \quad (55)$$

Remarks. Note this is of interest only when $\beta \ll 1$, since Proposition 3.6 bounds (w.h.p.) the number in question by $(1 + o(1))n^{t+1} q^t$; see Section 7.2 for a little more on the bounds in (55). The bound is also correct, but more trivial, when $t = 2$. The lemma doesn't actually require $S \subseteq G_0$: the proof shows that, for any $S \subseteq E(K_n)$ (of the stated size) with $\Delta_S = O(nq)$ (where Δ is maximum degree), we have the same bound for the number of P_t 's with terminal edges in S and internal edges in G_0 .

Preview. The proof of Lemma 2.2, which we are about to give, is based mainly on “coupling up”: using information about (G_0, F_0) to constrain what happens when we choose $G \setminus G_0$. (To this extent our strategy is similar to that of [6], but the resemblance ends there.) On the other hand, the proof of the crucial Lemma 7.1 in Section 7.1 is based on “coupling

down”: most of the work there is devoted to the proof of a similar statement (Lemma 7.3) involving only G (not G_0), from which the desired hybrid statement follows easily *via* coupling. In sum, we couple down to show that \mathcal{R} is likely (precisely, the conjunction of its failure with $\mathcal{Q} \wedge \mathcal{P}$ is unlikely), and couple up to show it is *unlikely*. A little more on the latter:

We would like to say that if G_0 is sufficiently nice—as it will be w.h.p.—then $\mathcal{P} \wedge \mathcal{R}$ is unlikely; this gives (51) *via* Lemma 7.1. The main point we need to add to Lemmas 7.1 and 7.2 is a deterministic one: if G_0 enjoys relevant genericity properties, together with the conclusion of Lemma 7.2, then, for each $S \subseteq G_0$, $R(S)$ is fairly small (depending on $|S|$; see (59)). Combined with $F \neq \emptyset$ (from \mathcal{P}), this will allow us to say that the lower bound on $|G \cap R(F_0)|$ ($= |F \cap R(F_0)|$) in \mathcal{R} is larger by a crucial factor $\alpha^{-\Omega(1)}$ than $|R(F_0)|p$ —its natural value when we “couple up”—which *ought* to make \mathcal{R} unlikely. But of course F_0 depends on G ; so, given G_0 , we are forced to sum the probability of this supposedly unlikely event over possible values S of F_0 , which turns out to mean that the whole argument would collapse if we were to replace the above $\alpha^{-\Omega(1)}$ by $\alpha^{-o(1)}$. (Here we again use \mathcal{P} , in this case to say α is small.)

A word on presentation. We prove the desired

$$\Pr(\mathcal{Q} \wedge \mathcal{P}) = o(1) \tag{56}$$

(= (51)) by producing a list of unlikely events and showing that at least one of these must hold if $\mathcal{Q} \wedge \mathcal{P}$ does. A more intuitive formulation might, for example, begin: “By Lemma 7.1 (since we assume $\mathcal{Q} \wedge \mathcal{P}$), *we may assume* \mathcal{R} .” But note this would really mean, not that we *condition* on \mathcal{R} —not something we can hope to understand—but that we need only bound probabilities $\Pr(\mathcal{S} \wedge \mathcal{R})$ for \mathcal{S} ’s of interest, and for a formal discussion this seems most clearly handled by something like the present approach.

For the derivation of Lemma 2.2 we need two more events (supplementing $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ above). The first of these is simply

$$\mathcal{S} = \{\alpha_0 \sim \alpha\}$$

(i.e. for any $\eta > 0$, $\alpha_0 = (1 \pm \eta)\alpha$ for large enough n ; recall α, α_0 were defined in (52)). The second, which we call \mathcal{T} , is the conjunction of a few properties of G_0 that we already know hold w.h.p., namely: $|G_0| \sim n^2 q/2$ (see Proposition 3.6); (27) and (28) for $l \in [\kappa-1, 2\kappa-6]$ (meaning, in view of (50), (28) if $l = \kappa-1$ and (27) otherwise); and the conclusion of Lemma 7.2 for $t \leq \kappa-1$ (actually we only need this for even t). We first outline and then fill in details.

We will show

$$\Pr(\mathcal{R} \wedge \{F \neq \emptyset\} \wedge \overline{\mathcal{S}}) = o(1). \quad (57)$$

(This is easy and a secondary use of \mathcal{R} . Note $\{F \neq \emptyset\}$ is implied by \mathcal{P} .)

We will also show that (*deterministically*)

$$\mathcal{R} \wedge \{F \neq \emptyset\} \wedge \mathcal{S} \implies |(G \setminus G_0) \cap R(F_0)| > .1\alpha n^2 p \quad (58)$$

provided ϑ is sufficiently small (this is again easy), and, as mentioned in the preview,

$$\mathcal{T} \implies |R(S)| = O(\alpha_S^{1+\delta} n^2) \quad (59)$$

for some fixed $\delta > 0$ and all $S \subseteq G_0$, where we set $\alpha_S = 2|S|/(n^2 q)$. Thus the conjunction of \mathcal{P} , \mathcal{R} , \mathcal{S} and \mathcal{T} implies (again, deterministically), the event—call it \mathcal{U} —that $|G_0| < n^2 q$ (say) and there is an $S \subseteq G_0$ (namely the one that will become F_0) satisfying (say):

$$\alpha_S < 2.1\lambda, |R(S)| = O(\alpha_S^{1+\delta} n^2), \text{ and } |(G \setminus G_0) \cap R(S)| > .09\alpha_S n^2 p. \quad (60)$$

Thus, finally, for (51) it is enough to show (by a routine calculation)

$$\Pr(\mathcal{U}) = o(1). \quad (61)$$

(Because: since $\overline{\mathcal{U}}$ implies $\overline{\mathcal{P}} \vee \overline{\mathcal{R}} \vee \overline{\mathcal{S}} \vee \overline{\mathcal{T}}$, (61) implies

$$\Pr(\mathcal{Q} \wedge (\overline{\mathcal{P}} \vee \overline{\mathcal{R}} \vee \overline{\mathcal{S}} \vee \overline{\mathcal{T}})) = \Pr(\mathcal{Q}) - o(1);$$

but the l.h.s. here is at most

$$\Pr(\mathcal{Q} \wedge \overline{\mathcal{P}}) + \Pr(\mathcal{Q} \wedge \mathcal{P} \wedge \overline{\mathcal{R}}) + \Pr(\mathcal{P} \wedge \mathcal{R} \wedge \overline{\mathcal{S}}) + \Pr(\overline{\mathcal{T}}) = \Pr(\mathcal{Q} \wedge \overline{\mathcal{P}}) + o(1)$$

(the second and third terms on the l.h.s. being bounded by Lemma 7.1 and (57) respectively); so we have $\Pr(\mathcal{Q} \wedge \mathcal{P}) = \Pr(\mathcal{Q}) - \Pr(\mathcal{Q} \wedge \overline{\mathcal{P}}) = o(1)$.)

Proof of (57). If $F \neq \emptyset$ (i.e. $\alpha > 0$) and \mathcal{R} holds, then $F \cap R(F_0) \neq \emptyset$, while by (36), for any $xy \in F \cap R(F_0)$,

$$|F| > \sigma(x, y) \geq \sigma_0(x, y) > .25n^{\kappa-2}q^{\kappa-1} = \Omega(\log n).$$

But then (since $\log n \gg p/q$) Proposition 3.15 says that w.h.p. $|F_0| \sim \vartheta|F|$, which is the same as \mathcal{S} . \square

Proof of (58). Note it is always true that $G_0 \cap R(F_0) \subseteq F_0$, since the endpoints of an $xy \in (G_0 \cap R(F_0)) \setminus F_0$ would be joined by a path (many paths) having odd intersection with F_0 , and adding xy to such a path would produce a C_κ having odd intersection with F_0 . (As mentioned earlier, this is the reason for “odd” in the definition of central.) So if \mathcal{R} , \mathcal{S} and $\{F_0 \neq \emptyset\}$ hold (and ϑ is slightly small) then

$$|(G \setminus G_0) \cap R(F_0)| > .12\alpha n^2 p - (1 + o(1))\alpha n^2 q/2 > .1\alpha n^2 p. \quad (62)$$

□

Proof of (59). Set $c = (\kappa - 3)/2$. For $l \in [c]$ and $\emptyset \neq S \subseteq G_0$ (for $S = \emptyset$ there is nothing to show), call an xy -path (S, l) -central if it is S -central and at least one of its S -edges is at distance l (along the path) from one of x, y . (So a path may be (S, l) -central for several l 's.) Let $\sigma_0(x, y; S, l)$ be the maximum size of a collection of internally disjoint (S, l) -central xy -paths in G_0 and

$$R_l(S) = \{\{x, y\} \in \binom{V}{2} : \sigma_0(x, y; S, l) > (.25/c)n^{\kappa-2}q^{\kappa-1}\}, \quad (63)$$

and notice that

$$R(S) \subseteq \cup_{l \in [c]} R_l(S). \quad (64)$$

Supposing temporarily (through (68)) that S and l have been specified, we abbreviate $\sigma_0(x, y; S, l) = \varsigma(x, y)$, $R_l(S) = R_l$ and use simply “rope” for “ $(S, 2l + 2)$ -rope” (defined before Lemma 7.1). Set $|R_l| = \rho_l n^2$ and

$$r = 2(\kappa - 1) - 2(l + 1) = 2(\kappa - l) - 4 \in [\kappa - 1, 2\kappa - 6]. \quad (65)$$

We next show that if G_0 satisfies

$$T := \max_{u,v} \tau^r(u, v) = O(n^{r-1}q^r) \quad (66)$$

(as implied by (27) and (28), so by \mathcal{T}), then

$$\text{the number of ropes is } \Omega(\rho_l n^{2l+3} q^{2l+2}). \quad (67)$$

Proof. Say a rope $P = (u_{l+1}, \dots, u_1, z, v_1, \dots, v_{l+1})$ is generated by $\{x, y\}$ if there are internally disjoint paths $(z, u_1, \dots, u_{\kappa-2}, w)$ and $(z, v_1, \dots, v_{\kappa-2}, w)$ with $\{z, w\} = \{x, y\}$. Each $\{x, y\} \in \binom{V}{2}$ generates at least $2^{\lfloor \varsigma(x, y)/2 \rfloor}$ such ropes (since a set of a internally disjoint (S, l) -central xy -paths, each with an S -edge at distance l from x , produces $\binom{a}{2}$ of them), while the number of

pairs generating a given rope is at most T (since in the scenario above, the complement of P in the cycle $(z, u_1, \dots, u_{\kappa-2}, w, v_{\kappa-2}, \dots, v_1, z)$ is a path of length r (see (65)) centered at w , so with P determines $\{x, y\}$). Thus the number of ropes is at least

$$T^{-1} \sum_{\{x,y\} \in R_l} 2^{\lfloor \varsigma(x,y)/2 \rfloor} = \Omega(|R_l|(n^{\kappa-2}q^{\kappa-1})^2/T) = \Omega(\rho_l n^{2l+3}q^{2l+2}). \quad \square$$

If we now also assume the conclusion of Lemma 7.2 for $t = 2l + 2$ (again, this is contained in \mathcal{T}), then combining that upper bound with the lower bound in (67) gives

$$\rho_l = O(\max\{\alpha_S^2, \alpha_S(nq)^{-l}\}) = O(\alpha_S^{1+\delta}), \quad (68)$$

with $\delta > 0$ depending only on κ . (Here we use $\alpha_S \geq n^{-2}$, valid since $S \neq \emptyset$.)

So, now letting l vary, it follows that if G_0 satisfies \mathcal{T} (and so all relevant instances of (66) and (55)), then (68) holds for all $l \in [c]$, which in view of (64) bounds $|R(S)|$ as in (59). \square

(It may be worth noting that for $l = 0$ the above argument gives only $\rho_l = O(\alpha_S)$, which loses the crucial δ in (68); thus the insistence on *central* paths in \mathcal{R} and Lemma 7.1.)

Proof of (61). Given G_0, S , we have $|(G \setminus G_0) \cap R(S)| \sim \text{Bin}(m, p')$, with $m \leq |R(S)|$ and $p' < p$ defined by $(1 - q)(1 - p') = 1 - p$ (as in (B) of Section 3.5). So for $|R(S)|$ as in (60), Theorem 3.2 gives

$$\Pr(|(G \setminus G_0) \cap R(S)| > .09\alpha_S n^2 p) < \exp[-\Omega(\alpha_S n^2 p \log(1/\alpha_S))],$$

where the implied constant depends on δ but not on ϑ . Thus, assuming $|G_0| < n^2 q$ (as given by \mathcal{U}), setting $\alpha_s = 2s/(n^2 q)$ (where s will be $|S|$, so $\alpha_s = \alpha_S$), and summing over $s < 2.1\lambda n^2 q$, we have

$$\begin{aligned} \Pr(\mathcal{U}|G_0) &< \sum_s \binom{n^2 q}{s} \exp[-\Omega(\alpha_s n^2 p \log(1/\alpha_s))] \\ &< \sum_s \exp[\alpha_s n^2 p \{(\vartheta/2) \log(2e/\alpha_s) - \Omega(\log(1/\alpha_s))\}], \end{aligned} \quad (69)$$

which is $o(1)$ for small enough ϑ (implying (61) since

$$\Pr(\mathcal{U}) = \sum \{\Pr(G_0) \Pr(\mathcal{U}|G_0) : |G_0| < n^2 q\}.$$

\square

7.1 Proof of Lemma 7.1

Fix $\varepsilon > 0$ (as in (50)) small enough to support the proofs of Propositions 7.5 and 7.8 below; these are our only constraints on ε , and it will be clear they are satisfiable. We continue to assume that p is as in (50).

Most of our effort here is devoted to proving the following variant of Proposition 7.1 in which we replace $\sigma_0(x, y, F_0)$ by $\sigma(x, y, F)$ and q by p .

Lemma 7.3. *W.h.p.*

$$G \in \mathcal{Q} \wedge \mathcal{P} \implies |\{xy \in F : \sigma(x, y, F) > .26n^{\kappa-2}p^{\kappa-1}\}| \geq .13\alpha n^2 p. \quad (70)$$

“Coupling down” will then easily get us to Lemma 7.1 itself. (The extra .01’s—relative to the pretty arbitrary .25 and .12 in (53), (54)—leave a little room for this.)

Preview. The proof of Lemma 7.3 breaks into two parts, roughly (w.h.p.): (a) if $G \in \mathcal{Q}$ (here we don’t need to assume $G \in \mathcal{P}$), then $\sigma(x, y)$ is close to its natural value for most $xy \in F$ (see the paragraph following the proof of Proposition 7.7); (b) a decent fraction of the paths produced in (a) are F -central (shown by limiting the number that are *not*; this is based on Proposition 7.8 and does assume $G \in \mathcal{P}$).

Definitions. It will be convenient to set

$$\Lambda = n^{\kappa-2}p^{\kappa-1},$$

since this quantity—essentially the typical number of paths in G joining a given pair of vertices—will appear repeatedly below. We write $Q \sim Q'$ when Q, Q' are distinct C_κ ’s sharing at least one edge. For edges e, f of G , we take

$$e \sim f \iff [\text{some } C_\kappa \text{ of } G \text{ contains both } e \text{ and } f], \quad (71)$$

$$e \approx f \iff [\text{there are } C_\kappa\text{'s } Q \sim Q' \text{ of } G \text{ with } e \in Q \text{ and } f \in Q'], \quad (72)$$

$S(e) = \{g \in G : e \sim g\}$, and $T(e) = \{g \in G : e \approx g\}$. For $\gamma \in (0, 1)$, let

$$L(\gamma) = \{\{x, y\} \in \binom{V}{2} : \sigma(x, y) < \gamma\Lambda\}$$

and $F(\gamma) = F \cap L(\gamma)$. Finally, with C as in Proposition 3.11 for $l = \kappa - 1$ (and, say, $\delta = 1/\kappa$), let \mathcal{S} be the event that G satisfies (23) so *not* the \mathcal{S} used above).

Fix $\zeta = .01$. Our goal in the next four propositions is to show that $F(1 - \zeta)$ is small, accomplishing (a) of our outline above. We do this by showing separately (in Propositions 7.6 and 7.7, using the tools provided by Propositions 7.4 and 7.5) that $F(\zeta)$ and $F(1 - \zeta) \setminus F(\zeta)$ are small.

Proposition 7.4. For $\gamma \in (0, 1)$ and distinct $\{x_1, y_1\}, \dots, \{x_c, y_c\} \in \binom{V}{2}$,

$$\Pr(\mathcal{S} \wedge \{\{x_i, y_i\} \in L(\gamma) \mid \forall i \in [c]\}) \leq n^{-(c-o(1))(\kappa/(\kappa-1))(1-\varepsilon)^{\kappa-1}\varphi(\gamma-1)}. \quad (73)$$

(Recall $\varphi(x)$ was defined in (14).) Note the bound here is natural, being, for p at the lower bound in (50) (and up to the $o(1)$), what Theorem 3.1 would give for the probability that c independent binomials, each of mean Λ , are all at most $\gamma\Lambda$.

Proof. Since \mathcal{S} gives $\tau(x, y) \leq \sigma(x, y) + C < (1 + o(1))\gamma\Lambda$ for $\{x, y\} \in L(\gamma)$, the event in (73) implies that $X := \sum_{i \in [c]} \tau(x_i, y_i) < (1 + o(1))c\gamma\Lambda$; so we just need to bound the probability of this.

In the notation of Theorem 3.3, with A_1, \dots, A_m the edge sets of the various $x_i y_i$ -paths (in K_n), we have $\mu \sim c\Lambda$ and $\overline{\Delta} = \mu + O(\Lambda^2/(np)) \sim \mu$. (If two of our paths, say P and Q , share $l \in [1, \kappa - 2]$ edges, then at least l internal vertices of P are vertices of Q ; so the contribution of such pairs to $\overline{\Delta}$ is less than

$$c^2 n^{2(\kappa-2)-l} p^{2(\kappa-1)-l} = O(\Lambda^2/(np)) = o(1)$$

(using the upper bound in (50) for the $o(1)$)). Thus Theorem 3.3 gives

$$\Pr(X < (1 + o(1))c\gamma\Lambda) < \exp[-(1 - o(1))\varphi(\gamma - 1)c\Lambda],$$

which, since $\Lambda > (1 - \varepsilon)^{\kappa-1}(\kappa/(\kappa-1)) \log n$, is less than the r.h.s. of (73). \square

Proposition 7.5. *W.h.p.*

$$\text{if } Q_1 \sim Q_2 \sim Q_3 \sim Q_4 \text{ are } C_\kappa \text{'s of } G \text{ then } |(\cup Q_i) \cap L(\zeta)| \leq 1. \quad (74)$$

Also, there is a fixed M such that w.h.p.

$$|S(e) \cap L(1 - \zeta)| < M \quad \forall e \in G. \quad (75)$$

(Note the Q_i 's in (74) need not be distinct.)

Proof. Write η_γ for the quantity $n^{-(1-o(1))(\kappa/(\kappa-1))(1-\varepsilon)^{\kappa-1}\varphi(\gamma-1)}$ appearing in (73) (here without the c).

Since \mathcal{S} occurs w.h.p., it suffices to show that the probability that it holds while either (74) or (75) fails is $o(1)$. Thus in the case of (74) we want to bound the probability that $\mathcal{S} \wedge \{J \subseteq G\} \wedge \{|J \cap L(\zeta)| \geq 2\}$ holds for

some $J \subseteq K_n$ of the form $\cup_{i \in [4]} Q_i$, where the Q_i 's are C_κ 's sharing edges as appropriate. With $\mathcal{T}(J) = \mathcal{S} \wedge \{|J \cap L(\zeta)| \geq 2\}$, this probability is at most

$$\begin{aligned} \sum \Pr(\{J \subseteq G\} \wedge \mathcal{T}(J)) &\leq \sum \Pr(J \subseteq G) \Pr(\mathcal{T}(J)) \\ &\leq O(n^{4\kappa-6} p^{4\kappa-3} \eta_\zeta^2) = o(1). \end{aligned}$$

Here the first inequality is an instance of Theorem 3.5 (since $\{J \subseteq G\}$ and $\mathcal{T}(J)$ are increasing and decreasing respectively), Proposition 7.4 gives $\Pr(\mathcal{T}(J)) = O(\eta_\zeta^2)$ (for any J), and the $o(1)$ holds (for small enough ε) since $n^{4\kappa-6} p^{4\kappa-3} = \tilde{\Theta}(n^{\kappa/(\kappa-1)})$. The argument for

$$\sum \Pr(J \subseteq G) = O(n^{4\kappa-6} p^{4\kappa-3}) \quad (76)$$

is similar to the proof of Proposition 3.11; briefly: if Q_1, \dots, Q_4 are C_κ 's, with $R_i = \cup_{j \leq i} Q_j$ and, for $i \geq 2$, $|E(Q_i) \setminus E(R_{i-1})| = b_i \leq \kappa - 1$ and $|V(Q_i) \setminus V(R_{i-1})| = a_i$, then $n^{a_i} p^{b_i} \leq \Lambda$ for $i \geq 2$ (since $b_i = a_i = 0$ or $b_i \geq a_i + 1$), which gives $n^{|V(R_4)|} p^{|E(R_4)|} \leq n^2 p \Lambda^4$ and (76).

Treatment of (75) is similar. Here J runs over subsets of K_n of the form $\cup_{i \in [M]} Q_i$, where the Q_i 's are C_κ 's with a common edge, and, with $\mathcal{T}(J) = \mathcal{S} \wedge \{|J \cap L(1 - \zeta)| \geq M\}$, the probability that \mathcal{S} holds while (75) fails is at most

$$\sum \Pr(\{J \subseteq G\} \wedge \mathcal{T}(J)) \leq O(n^2 p \Lambda^M \eta_{1-\zeta}^M) = o(1).$$

This is shown as above, with $n^{|V(J)|} p^{|E(J)|} \leq n^2 p \Lambda^M$ given by the passage following (76) (with M in place of 4) and the $o(1)$ valid for large enough M because $n^2 p \Lambda^M < n^{\kappa/(\kappa-1)} O(\log^{M/(\kappa-1)} n)$. \square

The next assertion is the only place where we use the condition $\{G \in \mathcal{Q}\}$ of (54) (and (51)).

Proposition 7.6. *W.h.p.*

$$G \in \mathcal{Q} \implies |F(\zeta)| = o(|F|). \quad (77)$$

Proof. By the first part of Proposition 7.5 it is enough to show that the r.h.s. of (77) follows (deterministically) from the conjunction of $\{G \in \mathcal{Q}\}$ and (74). But these imply that $|T(e) \cap F| \geq \zeta \Lambda$ for each $e \in F(\zeta)$: $\{G \in \mathcal{Q}\}$ gives at least one C_κ containing e ; this C_κ contains a second edge, xy , of F (since $F \in \mathcal{C}_\kappa^\perp$), which by (74) is not in $L(\zeta)$; and $T(e)$ contains at least $\zeta \Lambda$ (distinct) F -edges lying on xy -paths. Moreover, again by (74), $T(e) \cap T(f) = \emptyset$ for distinct $e, f \in F(\zeta)$. Thus $|F(\zeta)| < |F|/(\zeta \Lambda) (= o(|F|))$, as desired. \square

Proposition 7.7. *W.h.p.*

$$|F(1 - \zeta) \setminus F(\zeta)| = o(|F|). \quad (78)$$

Proof. It's enough to show that (75) implies (78) (since Proposition 7.5 says (75) holds w.h.p.). This is again easy: Set $B = F(1 - \zeta) \setminus F(\zeta)$ and consider the graph with vertex set F and adjacency as in (71). Each $e \in B$ has degree at least $\zeta\Lambda$ in this graph, while (75) says no vertex has more than M neighbors in B . Thus $|B|(\zeta\Lambda - M) \leq |F \setminus B|M$, which (since $\Lambda \gg 1$) gives (78). \square

Combining Propositions 7.6 and 7.7 completes part (a) of the preview following the statement of Lemma 7.3:

$$\text{w.h.p.} \quad G \in \mathcal{Q} \Rightarrow |F(1 - \zeta)| = o(|F|). \quad (79)$$

The next assertion, an echo of Section 3.3, provides technical support for part (b) (getting from (79) to Lemma 7.3 by controlling non- F -central paths).

For $v \in V$ and $S \subseteq \nabla_G(v)$, let $T_S(v)$ be the set of C_κ 's using two edges of S and $\tau_S(v) = |T_S(v)|$. (We could write simply T_S, τ_S , but keep the v as a reminder).

Proposition 7.8. *For each fixed $\theta > 0$ there exists C_θ such that w.h.p.: for all $v \in V$ and $S \subseteq \nabla_G(v)$, with $|S| = \gamma np$ and $\mu = \gamma^2 n^{\kappa-1} p^\kappa / 2$,*

$$\tau_S(v) < \begin{cases} (1 + \theta)\mu & \text{if } \gamma > \gamma_\theta := C_\theta \log \log n / \log n, \\ o(\mu/\gamma) & \text{in general.} \end{cases} \quad (80)$$

Proof. We first observe that there is a fixed B such that w.h.p. no v lies in more than B C_κ 's that meet $N(v)$ more than twice (basically because—here we omit the routine details—the expected number of such C_κ 's at a given v is $O(n^{\kappa-1} p^{\kappa+1}) = n^{-\Omega(1)}$). It is thus enough to prove Proposition 7.8 with T and τ replaced by T' and τ' , where $T'_S(v) = \{Q \in T_S(v) : |Q \cap N(v)| = 2\}$ and $\tau'_S(v) = |T'_S(v)|$.

Here we use a reduction similar to the one given by Proposition 3.11 (though we can't expect to do quite as well as in (23)). Let $\sigma_S(v)$ be the maximum size of a collection of C_κ 's from $T'_S(v)$ that are disjoint outside $\overline{N}(v) := \{v\} \cup N(v)$. Set $\psi(S) = \min\{|S|, \log^2 n\}$.

Proposition 7.9. *There exists D such that w.h.p. for all v and $S \subseteq \nabla_G(v)$,*

$$\tau'_S(v) - \sigma_S(v) < D\psi(S). \quad (81)$$

Proof. For fixed v and $S \subseteq \nabla_G(v)$, let $\Gamma = \Gamma_S$ be the graph on $T'_S(v)$ with $Q \sim R$ if Q and R share a vertex not in $\overline{N}(v)$. Since $\tau'_S(v) - \sigma_S(v) \leq |E(\Gamma)|$, (81) holds (for a suitable D) provided

- (i) the sizes of the components of Γ are $O(1)$ and
- (ii) the sizes of the induced matchings of Γ are $O(\psi(S))$;

so we would like to say that w.h.p. (i) and (ii) hold for all v and S . Here (and only here) we use $V(Q)$ for the set of vertices of Q not in $\overline{N}(v)$.

Of course (i) holds for all S (at v) iff it holds for $S = \nabla_G(v)$, so we just consider this case. Here we again (as in Proposition 3.11) want, for large enough M , (probable) nonexistence of $Q_1, \dots, Q_M \in T'_S(v)$ such that, for $i \geq 2$, $V(Q_i)$ meets, but is not contained in, $\cup_{j < i} V(Q_j)$. Arguing as for (25) we find that the total numbers, say a and b , of vertices (other than v) and edges used by such Q_1, \dots, Q_M satisfy

$$n^a p^b \leq n^{\kappa-1} p^\kappa (n^{\kappa-3} p^{\kappa-2})^{M-1}. \quad (82)$$

(Note here we do count neighbors of and edges at v . The bound says $n^a p^b$ is largest when each new Q_i meets what preceded it in a P_2 starting at v .) Since $n^{\kappa-1} p^\kappa = \Theta(np \log n)$ and $n^{\kappa-3} p^{\kappa-2} = \tilde{\Theta}(n^{-1/(\kappa-1)})$, the bound in (82) is $o(1/n)$ for slightly large M , as is the probability of seeing such Q_i 's at v .

For (ii), it will help to condition on $\nabla_G(v)$. Using ν' for the maximum size of an induced matching and invoking Proposition 3.6, we find that it's enough to show that, for a given v , $R \subseteq \nabla(v)$ of size less than $2np$ (say) and large enough D ,

$$\Pr(\exists S \subseteq R, \nu'(\Gamma_S) > D\psi(S) \mid \nabla_G(v) = R) = o(1/n). \quad (83)$$

So assume we have conditioned on $\{\nabla_G(v) = R\}$, with R as above. An easy verification (again similar to those in the proof of Proposition 3.11) gives, for any $S \subseteq R$ and, again, $\gamma_S = \gamma$ and $\Gamma_S = \Gamma$,

$$\begin{aligned} \tilde{\mu} = \tilde{\mu}_S &:= \mathbb{E}|E(\Gamma)| = O\left(\binom{|S|}{2} n^{\kappa-3} p^{\kappa-2} |S| n^{\kappa-4} p^{\kappa-3}\right) \\ &= O(\gamma^3 n^{2(\kappa-2)} p^{2(\kappa-1)}) = O(\gamma^3 \log^2 n); \end{aligned} \quad (84)$$

say $\tilde{\mu} < C\gamma^3 \log^2 n$ (with C fixed). On the other hand, with $\{Q_i, R_i\}$ the possible edges of Γ and $A_i = \{Q_i \cup R_i \subseteq G\}$, $\nu'(\Gamma) \geq l$ implies occurrence of some l independent A_i 's, an event whose probability (18) bounds by $(e\tilde{\mu}/l)^l$.

This leaves us with the union bound arithmetic. Here we first note that for $\nu'(\Gamma_S) < D \log^2 n \ \forall S$ we just need to check $S = R$, for which, in view of (84), we have $(e\tilde{\mu}/l)^l = o(1/n)$ for $l = D \log^2 n$ with a suitable D ($D > Ce$ is enough). We then need to say (again, for suitable D) that with probability $1 - o(1/n)$,

$$\nu'(\Gamma_S) < D|S| \text{ for all } S \text{ with } |S| < \log^2 n. \quad (85)$$

But with $s = \gamma np$, $\tilde{\mu} = \tilde{\mu}_s < C\gamma^3 \log^2 n$ and sums over $s \in [1, \log^2 n]$, the probability that (85) fails is at most

$$\sum \binom{|R|}{s} \left(\frac{e\tilde{\mu}}{Ds} \right)^{Ds} < \sum \exp \left[\gamma np \left\{ \log(2e/\gamma) + D \log \left(\frac{Ce\gamma^3 \log^2 n}{D\gamma np} \right) \right\} \right],$$

which, since we are in the range $\gamma np \in [1, \log^2 n]$, is easily $o(1/n)$. \square

We continue with the proof of Proposition 7.8, which, by Proposition 7.9, we now need only prove with $\tau_S(v)$ replaced by $\sigma_S(v)$. Here it will help to have a concrete $o(\cdot)$ in (80). Set $h = h(n) = (\log \log n)^{1/2}$ (we need $1 \ll h \ll \log \log n$) and, with C_θ (and thus γ_θ) TBA, set

$$K_\gamma = \begin{cases} 1 + \theta & \text{if } \gamma > \gamma_\theta, \\ (h\gamma)^{-1} & \text{otherwise.} \end{cases}$$

Given v and $S \subseteq \nabla_G(v)$ of size γnp (so we condition on $\{S \subseteq G\}$), and writing K for K_γ , we may apply Lemma 3.4 to obtain

$$\Pr(\sigma_S(v) > K\mu) < \begin{cases} \exp[-\theta^2 \mu/3] & \text{if } \gamma > \gamma_\theta, \\ \exp[-K\mu \log(K/e)] & \text{otherwise.} \end{cases} \quad (86)$$

Thus, with ξ_γ denoting the appropriate bound in (86), the probability of violating the σ_S -version of (80) with an S of size γnp is less than

$$n \binom{n}{\gamma np} p^{\gamma np} \xi_\gamma < \exp[\log n + \gamma np \log(e/\gamma)] \cdot \xi_\gamma \quad (87)$$

(where the terms preceding ξ_γ correspond to summing $\Pr(S \subseteq G)$ over $v \in V$ and $S \subseteq \nabla(v)$ of size γnp).

Finally, we should make sure the bound in (87) is small. Recalling (50), we have (for slightly small ε) $\Lambda > (1 - \varepsilon)^{\kappa-1} \kappa / (\kappa - 1) \log n > \log n$ and

$$\mu \ (= (\gamma^2 np/2) \Lambda) \ > (\gamma^2 np/2) \log n. \quad (88)$$

Thus for $\gamma > \gamma_\theta$ the bound in (87) is less than

$$\exp[\gamma np \cdot \{\log(e/\gamma) - \theta^2 \gamma \log n/6\} + \log n],$$

which is tiny ($\exp[-n^{\Omega(1)}]$) for fixed $C_\theta > 6\theta^{-2}$.

For $\gamma \leq \gamma_\theta$, noting that $(\gamma K_\gamma/2) \log(K_\gamma/e) \sim \log(1/\gamma)/(2h) = \omega(1)$ (and $\gamma np \geq 1$), and again using (88), we find that the r.h.s. of (87) is less than

$$\exp[\gamma np \cdot \{\log(e/\gamma) - (\gamma K_\gamma/2) \log(K_\gamma/e) \log n\} + \log n] = n^{-\omega(1)}.$$

And of course summing these bounds over γ gives what we want. \square

Proof of Lemma 7.3. Fix $\theta = .005$ and let $C = C_\theta$ and γ_θ be as in Proposition 7.8. Set $\gamma_v = d_F(v)/(np)$, and let φ_v be the number of C_κ 's of G using two F -edges at v . Let $\sigma^*(x, y)$ be the number of xy -paths having F -edges at one or both of x, y . Write \sum' and \sum'' for sums over v with $\gamma_v > \gamma_\theta$ and $\gamma_v \leq \gamma_\theta$ respectively. We have, w.h.p.,

$$\begin{aligned} \sum_{xy \in F} \sigma^*(x, y) &\leq 2 \sum_{v \in V} \varphi_v \\ &\leq n^{\kappa-1} p^\kappa \cdot [(1 + \theta) \sum' \gamma_v^2 + \sum'' o(\gamma_v)], \end{aligned} \quad (89)$$

where the first inequality comes from considering how many times each side counts the various C_κ 's of G , and the second is given by Proposition 7.8.

Since $\sum \gamma_v = \alpha n$, the second sum in (89) is $o(\alpha n)$. For the first, let $B = \{v \in V : \gamma_v > \theta\}$. If we now assume $\alpha = o(1)$ (as given by \mathcal{P}), then we have $|B| = o(n)$; so Proposition 3.7 (parts (a) and (b)) gives (w.h.p.)

$$|G[B]| \ll |B| \theta np < \sum_{v \in B} d_F(v) \leq \alpha n^2 p,$$

whence $\sum_{v \in B} \gamma_v np \leq 2|G[B]| + |\nabla_F(B)| < (1 + o(1)) \alpha n^2 p/2$,

$$\sum_{v \in B} \gamma_v < (1 + o(1)) \alpha n/2$$

and (recalling $d_F(v) \leq d_G(v)/2 \forall v$; see (9))

$$\sum_{v \in B} \gamma_v^2 \leq \max_v \gamma_v \sum_{v \in B} \gamma_v < (1 + o(1)) \alpha n/4. \quad (90)$$

Thus (since also $\sum_{v \in V \setminus B} \gamma_v^2 \leq \theta \sum_v \gamma_v = \theta \alpha n$) we find that the expression in square brackets in (89) is less than $(1/4 + 2\theta) \alpha n$, whence

$$\sum_{xy \in F} \sigma^*(x, y) \leq (1/4 + 2\theta) \alpha n^\kappa p^\kappa = .26 \alpha n^\kappa p^\kappa. \quad (91)$$

(To avoid confusion we note that the .26 here, which is more or less forced by the essentially tight bound in (90), is unrelated to the one in (70).)

Now let $F^* = \{xy \in F : \sigma(x, y) \geq (1 - \zeta) \Lambda\} (= F \setminus F(1 - \zeta))$. By (79), $|F^*| \sim \alpha n^2 p/2$, w.h.p. provided \mathcal{Q} holds. Note that (recall $\zeta = .01$)

$xy \in F^*$ has $\sigma(x, y; F) > .26\Lambda$ (as in (70)) unless $\sigma^*(x, y) > .73\Lambda$. (As noted earlier, xy -paths necessarily have odd intersection with F , so the only real requirement for such a path to be central is that it have an internal edge in F .) It thus follows from (91) that for $\tilde{F} := \{xy \in F^* : \sigma(x, y; F) \leq .26\Lambda\}$, we have

$$|\tilde{F}| \leq \frac{.26\alpha n^\kappa p^\kappa}{.73\Lambda} \leq .36\alpha n^2 p,$$

whence $|F^* \setminus \tilde{F}| \geq .13\alpha n^2 p$, implying (70). \square

Proof of Lemma 7.1. As mentioned earlier, Lemma 7.1 follows easily from Lemma 7.3 via “coupling down” (viewpoint (A) of Section 3.5): it is enough to show that if G satisfies the r.h.s. of (70) then w.h.p. it also satisfies \mathcal{R} ; that is, $|F \cap R(F_0)| \geq .12\alpha n^2 p$.

For $xy \in F' := \{xy \in F : \sigma(x, y; F) > .26\Lambda\}$ (see (70)), Theorem 3.1 gives

$$\Pr(\sigma_0(x, y; F_0) \leq .25n^{\kappa-2}q^{\kappa-1}) < \exp[-\Omega(n^{\kappa-2}q^{\kappa-1})] = n^{-\Omega(1)},$$

since members of a set of $\sigma(x, y; F)$ internally disjoint, F -central xy -paths survive in G_0 (and become F_0 -central) independently, each with probability $q^{\kappa-1}$. So by Markov’s Inequality, w.h.p.

$$|\{xy \in F' : \sigma_0(x, y; F_0) \leq .25n^{\kappa-2}q^{\kappa-1}\}| = o(|F'|).$$

The lemma follows. \square

7.2 Proof of Lemma 7.2

This is a simple consequence of Proposition 3.9, but for perspective a brief comment on the bounds may be helpful. The first bound—corresponding to a β^2 -fraction of all P_t ’s having their ends in S —is the generic value, and will be the truth if q is large enough that (w.h.p.) all $\tau^{t-2}(x, y)$ ’s are about the same. For smaller q one can sometimes do better by, e.g. (for even t), taking S to consist of all edges at distance $t/2 - 1$ from some small set of “centers,” producing something like the second bound.

Proof. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ be the eigenvalues of the adjacency matrix, A , of G_0 , with associated orthonormal eigenvectors v_1, v_2, \dots, v_n , say with $\max_j v_{1,j} > 0$. Let $M = A^{t-2}$ (so M has eigenvalues λ_i^{t-2} ($i \in [n]$), with eigenvectors v_i), and $f = (d_S(x) : x \in V) = \sum \beta_i v_i$.

The number of (S, t) -ropes may w.h.p. be bounded by

$$\begin{aligned}
fMf^T &= \sum \lambda_i^{t-2} \beta_i^2 \\
&\leq \lambda_1^{t-2} \beta_1^2 + \max\{\lambda_2, |\lambda_n|\}^{t-2} \|f\|_2^2 \\
&< (1 + o(1))[(nq)^{t-2} \beta_1^2 + (4nq)^{(t-2)/2} \|f\|_2^2], \tag{92}
\end{aligned}$$

where we used $\sum \beta_i^2 = \|f\|_2^2$ and the second inequality is given by (20). We then need bounds on β_1^2 and $\|f\|_2^2$, both of which are easy: w.h.p.

$$\beta_1 = \langle f, v_1 \rangle \sim n^{-1/2} \sum d_S(v) = 2n^{-1/2} |S| = \beta n^{3/2} q$$

(using (21)) and

$$\|f\|_2^2 = \sum d_S^2(x) \leq \Delta_S \sum_x d_S(x) < (1 + o(1)) nq \cdot 2|S| \sim \beta n^3 q^2.$$

The lemma follows. \square

8 Proof of Theorem 1.5

In what follows we set $\mathcal{E}(K_n) = \mathcal{E}$, $\mathcal{C}_H(K_n) = \mathcal{C}_H$ and so on. We prove (sketchily) Theorem 1.5 for $n \geq v_H + 2$ —which is best possible e.g. if $H = K_\kappa$ with $\kappa \geq 4$ (e.g. since for $n \leq \kappa + 1$, $\mathcal{C}_H^\perp \supseteq \mathcal{C} \cap \mathcal{D}$)—and add a note at the end to cover $H = C_\kappa$ and $n \geq \kappa$.

We first note that $\mathcal{C}_H = \mathcal{E}$ if $|H| = 1$ (trivially) and $\mathcal{C}_H = \mathcal{D}$ if $|H| = 2$. (Since each of P_2 , $2K_2$ (a 2-edge matching) is the sum of two copies of the other, the copies of an H of size 2 span all 2-edge subgraphs, and so all even subgraphs, of K_n .) Moreover, if H is a matching then \mathcal{C}_H is easily seen to contain (all copies of) K_2 if $|H|$ is odd or $2K_2$ if $|H|$ is even, so is equal to \mathcal{E} or \mathcal{D} as appropriate.

We may thus restrict attention to H containing a vertex x of degree at least 2, and observe that in this case $\mathcal{C}_H \supseteq \mathcal{C}_4$. (The sum of two copies of H that differ only in the copy of x is a $K := K_{2,d(x)}$, and repeating this with K and one of its divalent vertices produces a C_4 .)

Since $\mathcal{C}_4 = \mathcal{C} \cap \mathcal{D}$, we're done if H is even Eulerian. Otherwise let \tilde{H} be a copy of H in K_n and F a smallest element of $\tilde{H} + \mathcal{C}_4$. Then F clearly belongs to the same case of (5) as H and we claim it is either a triangle or the disjoint union of a matching and star (so possibly just a matching or just a star). Note this is enough, as the copies of F are then easily seen to generate the desired subspace of \mathcal{E} : if H is Eulerian then $F = K_3$; otherwise

we may add two copies of F to produce a P_2 , so the generated space contains \mathcal{D} . (Minor note: $|V(F)| \leq |V(H)| + 1$ since all odd vertices of F must also be odd in \tilde{H} .)

For the claim we observe that F cannot contain a P_3 (since adding a C_4 containing such a P_3 reduces $|F|$); disjoint P_2 's (reduce by adding a C_6); or $K_3 + K_2$ (convert to P_4 , then reduce to P_2).

Finally, for $H = C_\kappa$ and $n \geq \kappa \geq 4$ (for $\kappa = 3$ there is nothing to show), it is enough to observe that the sum of two copies of H on the same vertex set and sharing a $P_{\kappa-3}$ is a C_4 ; so $\mathcal{C}_H = \mathcal{C} \cap \mathcal{D}$ if κ is even, while for odd κ , $\mathcal{C} \cap \mathcal{D} \subset \mathcal{C}_H \subseteq \mathcal{C}$ implies $\mathcal{C}_H = \mathcal{C}$.

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